COSETS OF SUBGROUPS

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Reference: *A Gentle Introduction to Group Theory*, Bana Al Subaiei & Muneerah Al Nuwairan, Section 7.4.

Given a group G, an element $a \in G$ and one of the subgroups of G, called H, we can define the set of left cosets of a as the set of elements formed by left-multiplying each element in H by a. In formal notation, the left cosets are defined by

$$aH = \{a * h : h \in H\} \quad \text{for all } a \in G \tag{1}$$

Similarly, we can define the right cosets by right-multiplying H by a:

$$Ha = \{a * h : h \in H\} \quad \text{for all } a \in G \tag{2}$$

Note that a coset is not necessarily a group or subgroup, as it may not contain an identity element.

Example 1. Consider the group of integers $G = (\mathbb{Z}, +)$ under addition, with identity element 0, and take the subset of $\mathbb{Z}$ formed by positive and negative multiples of 6, that is the subset $H = (6\mathbb{Z}, +)$. The left cosets of H are found by adding integers to the left of a multiple of 6, so the elements of such a left coset are $\ldots, -2 + 6\mathbb{Z}, -1 + 6\mathbb{Z}, 6\mathbb{Z}, 1 + 6\mathbb{Z}, 2 + 6\mathbb{Z}, \ldots m + 6\mathbb{Z}, \ldots$. However, we can write an integer m using the quotient remainder theorem as

$$m = 6q + r \tag{3}$$

where $0 \leq r < 6$. Therefore a left coset has the form

$$6q + r + 6\mathbb{Z} = r + 6\mathbb{Z} \tag{4}$$

so there are only 6 distinct values of r, namely $r = 0, 1, 2, 3, 4, 5$. Thus the subset H has only 6 distinct left cosets. In this case, adding m to the right of an element in $\mathbb{Z}$ has the same effect as adding it to the left, so the left and right cosets are the same.

Example 2. Consider the symmetric group of order 4, $\mathfrak{S}_4$, which has $4! =$ 24 elements. This has 30 subgroups (found from Maple), so we'll choose one of them, namely the subgroup generated by the element $(1,4,3)$:

$$H = \langle (1,4,3) \rangle = \{e, (1,3,4), (1,4,3)\} \tag{5}$$

We can see that this is a subgroup since it contains the identity e and $(1,3,4)$ is the inverse of $(1,4,3)$. In principle, to find the right cosets we'd need to right-multiply each element in H by one of the 24 elements in $\mathfrak{S}_4$. However, it turns out that there are only 8 distinct right cosets (this is a consequence of Lagrange's theorem, which we'll study later). Using Maple, we find that the right cosets are (I've omitted the commas between numbers to avoid clutter):

$$\{(143), (134), ()\} \tag{6}$$
$$\{(1432), (1324), (23)\} \tag{7}$$
$$\{(124), (12)(34), (123)\} \tag{8}$$
$$\{(14)(23), (243), (132)\} \tag{9}$$
$$\{(1342), (24), (1423)\} \tag{10}$$
$$\{(14), (34), (13)\} \tag{11}$$
$$\{(142), (13)(24), (234)\} \tag{12}$$
$$\{(1243), (1234), (12)\} \tag{13}$$

As an example of how these sets are generated, consider the line 7. We can generate this by right-multiplying H by $(2,3)$ and remembering that in permutations we work from right to left. We get

$$e\,(23) = (23) \tag{14}$$
$$(134)\,(23) = (1324) \tag{15}$$
$$(143)\,(23) = (1432) \tag{16}$$

The other lines are generated similarly. From top to bottom, we right-multiply by e, (23), $(12)\,(34)$, (243), (24), (34), (234) and (12). These 8 elements from $\mathfrak{S}_4$ are all that are needed to generate the 8 distinct right cosets. The other elements from $\mathfrak{S}_4$ all produce one of these 8 cosets. For example, if we right-multiply by (14), we get

$$e\,(14) = (14) \tag{17}$$
$$(134)\,(14) = (34) \tag{18}$$
$$(143)\,(14) = (13) \tag{19}$$

This reproduces 11 above.

We can generate the left cosets in a similar way and we get

$$\{(124), (13)(24), (243)\} \tag{20}$$
$$\{(1243), (1324), (24)\} \tag{21}$$

$$\{(14),(34),(13)\} \tag{22}$$

$$\{(14)(23),(234),(123)\} \tag{23}$$

$$\{(1432),(1342),(12)\} \tag{24}$$

$$\{(143),(134),()\} \tag{25}$$

$$\{(1423),(1234),(23)\} \tag{26}$$

$$\{(142),(12)(34),(132)\} \tag{27}$$

In this case, the rows are generated by left-multiplying H by, in order, (243), (24), (34), (234), (12), e, (23) and $(12)(34)$. Note that the set of left cosets is different from the set of right cosets. This is because permutation groups are non-abelian (they don't commute), except for multiplying by the identity e.

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